Q) If p is an odd prime and a, b are coprime then show that,

$$\gcd\left(\frac{a^p+b^p}{a+b}, a+b\right) \in \{1, p\}$$

Ans:- $a^p+b^p = (a+b)(a^{p-1} - a^{p-2}b + \cdots + b^{p-1})$

$$\frac{a^p+b^p}{a+b} = a^{p-1} - a^{p-2}b + \cdots + b^{p-1} =$$

$$\gcd\left(\frac{a^p+b^p}{a+b}, a+b\right) = \gcd(a^{p-1} - a^{p-2}b + \cdots + b^{p-1}, a+b)$$

$$= \gcd((a+b)(\cdot\ ) + pb^{p-1}, a+b) = \gcd(pb^{p-1}, a+b)$$

$a^{p-1} + a^{p-2}b + \cdots + b^{p-1}$

$= (a+b)(a^{p-2} - 2a^{p-3}b + 3a^{p-4}b^2 - \cdots - (p-1)b^{p-2}) + pb^{p-1}$

$= a^{p-1} - 2a^{p-2}b + 3a^{p-3}b^2 - \cdots - (p-1)ab^{p-2}$

$= a^{p-1} - 2a^{p-2}b + 3a^{p-3}b^2 + \cdots + (p-2)ab^{p-2} - (p-1)b^{p-1} + pb^{p-1}$
$\quad + a^{p-2}b - 2a^{p-3}b^2 + \cdots - ab^{p-2} + b^{p-1}$

$= a^{p-1} - a^{p-2}b + a^{p-3}b^2 - \cdots - ab^{p-2} + b^{p-1}$

$$\gcd(pb^{p-1}, a+b) = \gcd(p, a+b) = \{1, p\}$$

$\Rightarrow a+b$ and b coprime
$\Rightarrow a+b$ and b^{p-1} coprime

Q) Let $n, p > 1$ be positive integers and p be a prime. Given that $n | (p-1)$ and $p | (n^3-1)$ prove that $4p-3$ is a perfect square

Ans.- $n | (p-1) \Rightarrow (p-1) = nk \Rightarrow nk+1 = p \Rightarrow p \geq n+1$
$\Rightarrow p \nmid (n-1)$

$p | (n^3-1) \Rightarrow (nk+1) | (n-1)(n^2+n+1)$

$\Rightarrow (nk+1) | (n^2+n+1) \longrightarrow \Rightarrow p \leq |n^2+n+1|$

but $n^2+n+1 \geq 0$

$\Rightarrow p \leq n^2+n+1$ — (1)

$\Rightarrow (nk+1) | (k(n^2+n+1))$

$\Rightarrow (nk+1) | (kn^2+kn+k)$

$\Rightarrow (nk+1) | (kn^2+kn+k - n(nk+1))$

$\Rightarrow (nk+1) \mid (kn^2+kn+k)$ ①

$\Rightarrow (nk+1) \mid (kn^2+kn+k - n(nk+1))$

$\Rightarrow (nk+1) \mid (nk+k-n)$

$\Rightarrow nk+1 \le nk+k-n \Rightarrow n+1 \le k \Rightarrow n(n+1)+1 \le nk+1 = p$

$\Rightarrow n^2+n+1 \le p$ — ②

From ① & ② :– $p = n^2+n+1$

$\Rightarrow 4p-3 = 4n^2+4n+4-3 = 4n^2+4n+1 = (2n+1)^2$ = perfect square ($2n+1 \in \mathbb{Z}$)

Q) (IMO 1959) Prove that for any natural number n, the fraction $\dfrac{21n+4}{14n+3}$ is irreducible.

Ans :– $\gcd(21n+4, 14n+3) = \gcd(7n+1, 14n+3)$ [$7n+1 = 21n+4-14n-3$] $= \gcd(7n+1, 14n+3-2(7n+1))$

$= \gcd(7n+1, 1) = 1$

$\Rightarrow \gcd(21n+4, 14n+3) = 1$

Q) <u>Home Work</u> :– Show that for all primes p, $Q(p)$ is an integer where,

$$Q(p) = \prod_{k=1}^{p-1} \left(k^{2k-p-1}\right)$$